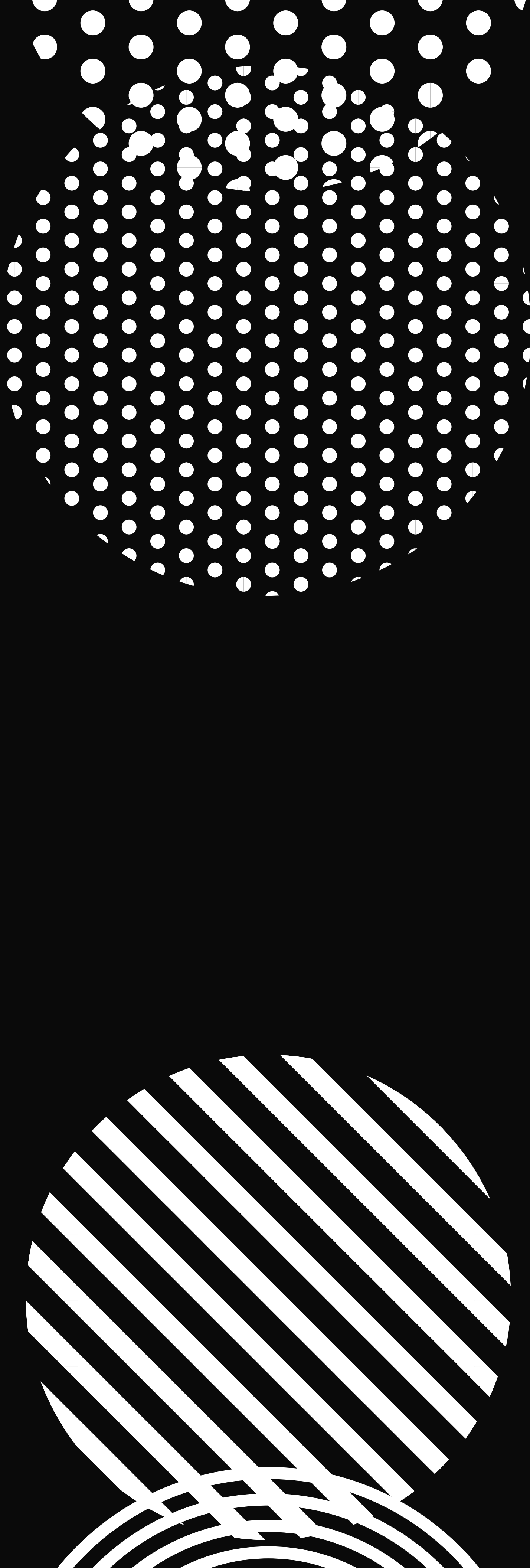


## §0 Last time

<u>Thm</u> (Cantor): For all $X$, $\#2^X > \#X$.

<u>Pf</u>: Suppose $F: X \to 2^X$ is a surj.

Then for $x \in X$, $F(x)$ is a function

$X \to \{0,1\}$ which we denote by $f_x$. Define an element $g : X \to \{0, 1\}$ of $2^X$

as follows

$$g(x) := 1 - f_x(x)$$

Note then that $g \neq f_x$ for

any $x \in X$ as $g(x) = 1 - f_x(x) \neq f_x(x)$.

Thus, $g \in 2^X - F(X)$, $\blacksquare$

<u>NB:</u> Recall that there is a bij.

$P(X) \longrightarrow 2^X$ given by $S \longmapsto \mathbb{1}_S$ where

$$\mathbb{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Recall from last time we showed that for a surj. $F: X \longrightarrow P(X)$, and we built a weird set

$$S = \{x \in X : x \notin F(x)\} \in \mathcal{P}(X)$$

in $\mathcal{P}(X) - F(X)$. The relationship betw this $S$ and the above $g$ is

$$g = \mathbb{1}_S$$

∎

# § 1 Cardinality of $\mathbb{R}$ and CH

Exercise: go read proof!

**Prop:** $\#\mathbb{R} = 2^{\aleph_0} := \# P(\mathbb{N})$.

**Pf:** By Schroeder-Bernstein theorem STS that $\#\mathbb{R} \leq 2^{\aleph_0}$ and $2^{\aleph_0} \leq \#\mathbb{R}$.

**$\#\mathbb{R} \leq 2^{\aleph_0}$:** Note that as $\#\mathbb{Q} = \aleph_0$

then $\#P(\mathbb{Q}) = 2^{\aleph_0}$. So, STS $\exists$ inj. $\mathbb{R} \to P(\mathbb{Q})$. But such an inj. is

$$\mathbb{R} \to P(\mathbb{Q}), \quad r \longmapsto \{q \in \mathbb{Q} : q < r\}.$$

<u>$2^{\aleph_0} \leq \#\mathbb{R}$:</u> Define

$$2^{\mathbb{N}} \longrightarrow \mathbb{R}, \quad (f : \mathbb{N} \to \{0,1\}) \longmapsto \sum_{n=0}^{\infty} \frac{f(n)}{3^{n+1}}$$

This is injective: assume $f \neq g$, WTS:

$$\sum_{n=0}^{\infty} \frac{f(n)}{3^{n+1}} \neq \sum_{n=0}^{\infty} \frac{g(n)}{3^{n+1}}$$

Let $k = \min\{n : f(n) \neq g(n)\}$. WLOG $f(k)=1$ and $g(k)=0$.

Then,

$$\sum_{n=0}^{\infty} \frac{f(n)}{3^{n+1}} - \sum_{n=0}^{\infty} \frac{g(n)}{3^{n+1}} = \frac{1}{3^{k+1}} - \sum_{n=k+1} \frac{g(n)-f(n)}{3^{n+1}}$$

$$\geq \frac{1}{3^{k+1}} + \sum_{n=k+1} \frac{1}{3^{n+1}}$$

$$= \frac{1}{3^{k+1}} - \frac{1}{3^{k+1}} \cdot \frac{1}{1-\frac{1}{3}}$$

$$= \frac{1}{3^{k+1}} - \frac{1}{2\, 3^{k+1}}$$

$$\geq 0 \qquad \blacksquare$$

So, by Cantor's theorem

$$\beth_0 = \aleph_0 < 2^{\aleph_0} =: \beth_1 < 2^{\beth_1} = \beth_2 < \cdots$$

$$\begin{gathered} \| \\ \#\mathbb{R} \end{gathered}$$

But, also

$$\aleph_0 < \aleph_1 := \text{smallest cardinal bigger than } \aleph_0 < \aleph_2 := \ldots$$

<u>Continuum hypothesis</u>: $\aleph_1 = \beth_1 = \#\mathbb{R}$

<u>Thm</u> (Cohen): If ZFC are the usual axioms of set theory ("=" math) then $\exists$

1) Some axioms $A$ s.t. in $ZFC \cup A$ the CH is true,

2) Some axioms $B$ s.t. in $ZFC \cup B$ the CH is false.

# §3 Transcendental numbers

<u>Definition:</u> the set $\bar{\mathbb{Q}}$ of <u>algebraic numbers</u> is

$$\bar{\mathbb{Q}} = \left\{ x \in \mathbb{C} : \begin{array}{l} x \text{ is the root} \\ \text{of a non-zero} \\ \text{poly w/ rational coeffs} \end{array} \right\}$$

e.g.:

- $\mathbb{Q} \subseteq \overline{\mathbb{Q}}$
- $\sqrt{2} \in \overline{\mathbb{Q}}$, root of $x^2 - 2$
- $\cos\left(\frac{\pi}{7}\right) \in \overline{\mathbb{Q}}$, root of $8x^3 - 4x^2 - 4x + 1$

**Definition:** The set of **transcendental** numbers

$$\mathbb{T} := \mathbb{C} - \overline{\mathbb{Q}}$$

**Q:** Is $\mathbb{T}$ empty?

**A:** No — but it is really hard to prove any number you suspect is in $\mathbb{T}$ is in $\mathbb{T}$.

e.g.: (Hermite, 1873): $e$ is transcendental

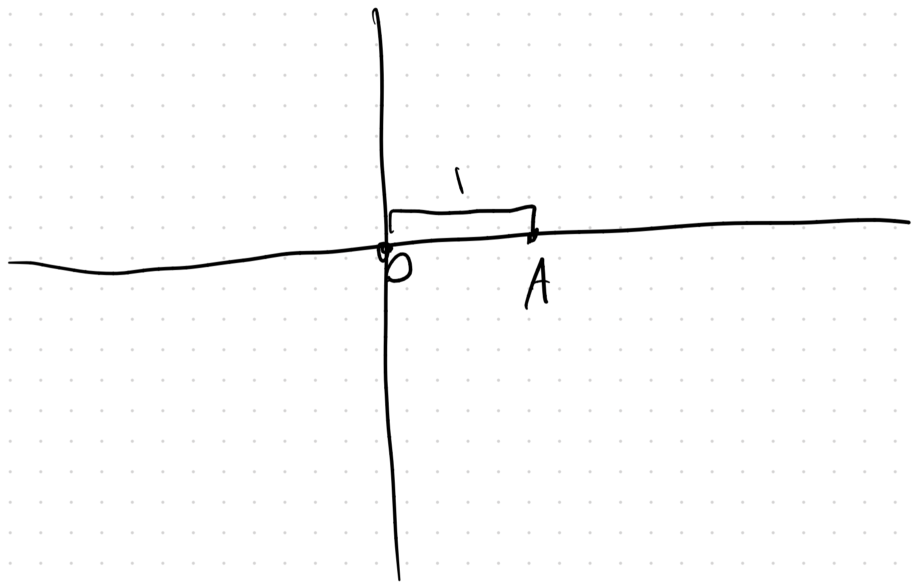
e.g.: (Lindemann, 1882): $\pi$ is transcendental

Q (5th century BC): Is it possible to construct a square whose area is the same as the unit circle?

— i.e., is it possible to construct a

line segment of length $\sqrt{\pi}$?

Construct: Start w/

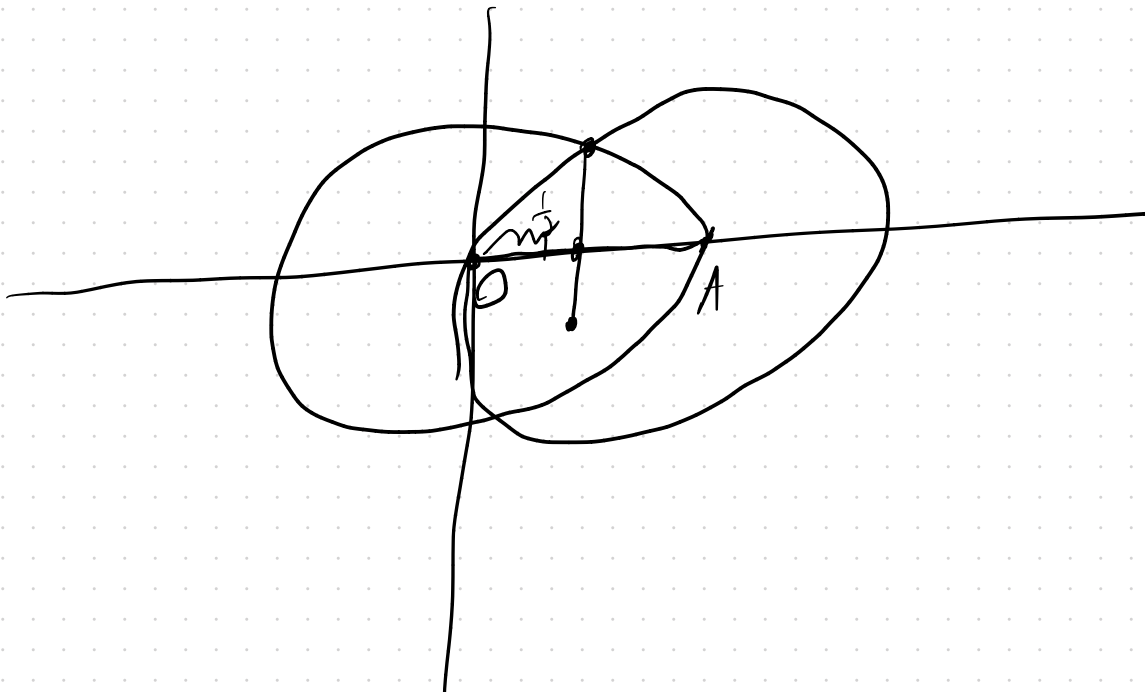



a) an idealized straightedge

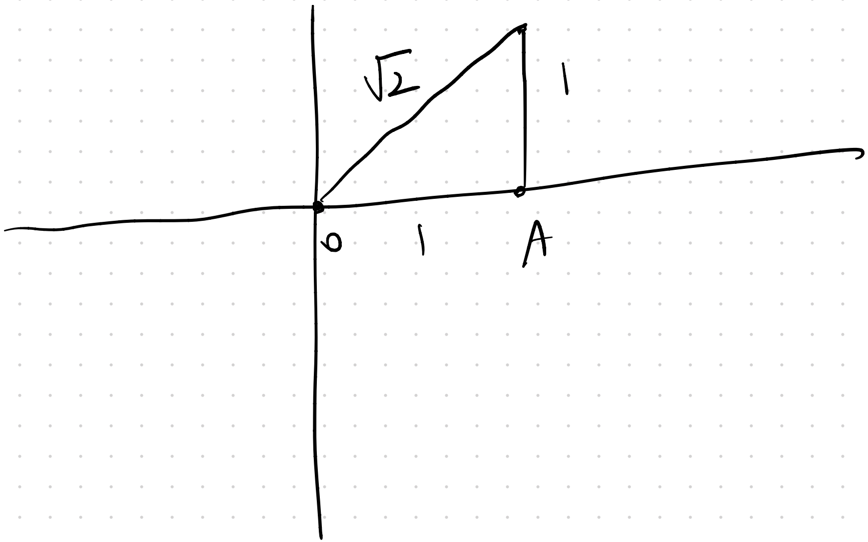
+ b) an idealized compass

a line segment is constructible if you can make it using this setup in finitely many steps

e.g.) $\frac{1}{2}$ is constructible



e.g. $\sqrt{2}$ is constructible




<u>Thm</u> (Lindemann, 1882): You cannot square the circle

<u>Pf</u>: <u>Step 1</u> (Hermite's thm, 1873): $e$ is transcendental

<u>Step 2</u> (Lindemann - Weierstrass thm, 1882): If $\alpha$ is an algebraic number then $e^{\alpha}$ is transcendental

<u>Step 3</u> (Euler, 1740): $e^{i\pi} = -1$

<u>Step 4</u> (Step 2 + Step 3): If $\pi$ was algebraic then $i\pi$ is algebraic, so by Lindemann-Weierstrass

$e^{i\pi}$ is transcendental but $e^{i\pi} = -1$ and $-1$ is

not transcendental. Contradiction.

Step 5 (Galois – Wantzel, 1837): A number

$x \in \mathbb{C}$ is constructible iff it can be

obtained from elements of $\mathbb{Q}$ by iterated

applications of $+, -, *, \div, \sqrt{\ }$

Step 6: All these constructible numbers are algebraic so as $\pi \notin \bar{\mathbb{Q}}$, $\sqrt{\pi} \notin \bar{\mathbb{Q}}$

So $\sqrt{\pi}$ is not constructible. $\blacksquare$

Open question: Is $e + \pi \in \bar{\mathbb{Q}}$?

__Thm:__ $\overline{\mathbb{Q}}$ is countable, but $\mathbb{C}$ is uncountable, ergo "most" numbers are transcendental.

__Lemma 1:__ Let $X$ be a set and $\{S_i\}_{i \in I}$ be a collection of subsets such that $I$ and each $S_i$ is countable. Then, $\bigcup_{i \in I} S_i$ is countable.

Pf: As $I$ is countable there is a surj. $g: \mathbb{N} \to I$ and as each $S_i$ is countable $\forall i$ $\exists$ are surjections $f_i: \mathbb{N} \to S_i$. Consider

$$F: \mathbb{N} \times \mathbb{N} \to \bigcup_i S_i$$

$$(m,n) \longmapsto f_{g(m)}(n)$$

This is surj: let $x \in \bigcup_{i \in I} S_i$. Then $x \in S_{i_0}$ for some $i_0$. As $g$ is surj. $\exists\ m_0 \in \mathbb{N}$ s.t. $g(m_0) = i_0$, and as $f_{i_0}: \mathbb{N} \to S_{i_0}$ is surj $\exists\ n_0 \in \mathbb{N}$ s.t. $f_{i_0}(n_0) = x$. Then

$$F(m_0, n_0) = f_{g(m_0)}(n_0) = f_{i_0}(n_0) = x.$$

But, we proved that $\mathbb{N} \times \mathbb{N}$ is countable, thus $S_0$ is $\bigcup_{i \in F} S_i$ $\boxed{\equiv}$

<u>Lemma 2:</u> For all $n \geq 1$, $\# \mathbb{Q}^n = \aleph_0$.

<u>Pf:</u> We proceed by induction.

<u>Base Case $(n=1)$:</u> $\mathbb{Q}$ is countable $\Rightarrow \exists$ bij

$t : \mathbb{Q} \to \mathbb{N}$.

<u>Base Case $(n=2)$:</u>

Observe we have already shown that $\exists$ bij

$$\mathbb{Q}^2 \xrightarrow{(t,t)} \mathbb{N}^2 \xrightarrow{\text{constructed previously}} \mathbb{N} \xrightarrow{t} \mathbb{Q}$$

IH: Assume $\# \mathbb{Q}^n = \aleph_0$. Then observe

$\exists$ bij

$$\mathbb{Q}^{n+1} = \mathbb{Q}^{n-1} \times \mathbb{Q}^2 \xrightarrow[n=2]{BC} \mathbb{Q}^{n-1} \times \mathbb{Q} = \mathbb{Q}^n$$

So $\# \mathbb{Q}^{n+1} = \# \mathbb{Q}^n = \aleph_0$ $\blacksquare$

Pf of Thm: Note

$$R(a_0, \dots, a_n) \\ \overline{\mathbb{Q}} = \bigcup_{n \in \mathbb{N}} \bigcup_{(a_0, \dots, a_n) \in \mathbb{Q}^{n+1}} \left\{ \begin{array}{c} \text{roots of} \\ a_0 + a_1 x + \dots + a_n x^n \end{array} \right\}$$

As $\mathbb{Q}^{n+1}$ is countable by Lemma 2, and $R(a_0, \dots, a_n)$ we deduce from Lemma 1 that

$$S_n = \bigcup_{(a_0, \dots, a_n) \in \mathbb{Q}^{n+1}} R(a_0, \dots, a_n)$$

is countable for all $n$. So, as $\mathbb{N}$ is countable and each $S_n$ is countable

$$\bar{\mathbb{Q}} = \bigcup_{n \in \mathbb{N}} S_n$$

is again countable by Lemma 1 $\square$

**Thm:** $\mathbb{T} \subseteq \mathbb{C}$ is uncountable.

**pf:** Assume not, then as

$$\mathbb{C} = \bar{\mathbb{Q}} \cup \mathbb{T}$$

and $\bar{\mathbb{Q}}$ is countable we see by Lemma 1 that $\mathbb{C}$ is countable.

By, $\mathbb{C} \supseteq \mathbb{R}$. Contradiction